# Numerical Study of the Two-Dimensional-Navier-Stokes-α Model of Turbulence

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The Navier-Stokes equations (NSE) are thought to provide a complete mathematical description of fluid turbulence. Because of the great difficulty in obtaining general analytical solutions for turbulent flows, we look to the capabilities of computers to perform direct numerical simulation (DNS) of the NSE in order to make progress.

Due to the nonlinearity of the NSE, there is a cascade of turbulence, from large scales- where forces due to stirring are introduced, to small scales- where viscous forces start to dominate. Thus, for very turbulent flows, we need to compute the NSE over a wide range of scales when performing DNS. This incurs a computational cost that is prohibitive even with the increasingly powerful computers that are available today.

It is therefore important to find a turbulence model that we can compute more efficiently than the NSE, and which recovers as much of the characteristic features of turbulence as possible. The purpose of this project is to validate the accuracy of one such model, the two-dimensional (2D) Navier-Stokes-α (NS-α) turbulence model. (see [1] for a complete mathematical theory of this model).

How do we verify the accuracy of this model? Since the NSE correctly predicts the characteristic features of a turbulent flow, we would like to get good agreement of the numerical calculation of the 2D NS-α model with DNS of 2D NSE in some range of scales of interest. Based on analytical results of [2], we expect that the energy spectrum of the NS-α will not deviate from the energy spectrum of NSE for scales of motion greater than the lengthscale parameter α. That is, when using the NS-α to simulate turbulence, we expect to retain the true statistics of the flow for scales of motion greater than the lengthscale parameter α. In Figure 1, we illustrate what we expect to see, as a function of wavenumber $k$. The wavenumber $k$ is the reciprocal of a wavelength or sometimes referred to as spatial frequency. The schematic shows that the NSE spectrum (blue) should coincide with the NS-α spectrum (magenta) up to wavenumber $k_\alpha$. This is equivalent to retaining the statistics of the flow for lengthscales greater than α.

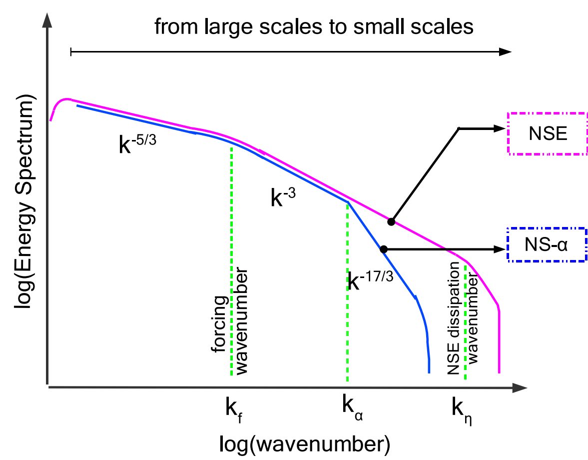


*Figure 1: A (log-log) sketch of what we expect the energy spectrum for the 2D NS-α would look like in comparison to DNS.*

What makes this model more computationally favorable than DNS? The NS-α model of turbulence modifies the nonlinearity in the NSE to prevent the cascading of turbulence at scales smaller than the parameter α while correctly predicts the statistics of the large scales of lengthscale bigger than α. (see [3] for a more complete physical description.) Thus, the range of scales of motion that we will need to compute now depends on the lengthscale parameter α. This makes the NS-α model of turbulence more computable since we do not have to resolve as much of the smaller scales (those scales which corresponds to wavenumbers greater than $k_\alpha$) in order to obtain the statistics of a turbulent flow for scales less than $\alpha$.

Our preliminary numerical results exhibit the expected behavior of this model. In Figure 2, we compute the 2D NS-$\alpha$ in a periodic box. We compute the energy spectrum of the the NS-$\alpha$ when different values of the length-scale $\alpha$ parameter are used for a given fixed resolution and fixed viscosity. For values of $\alpha$ corresponding to wave numbers $k_\alpha = 17, k_\alpha = 30$ and $k_\alpha = 166$ in a $2048^2$ grid simulation, the NS-$\alpha$ model performed as expected.

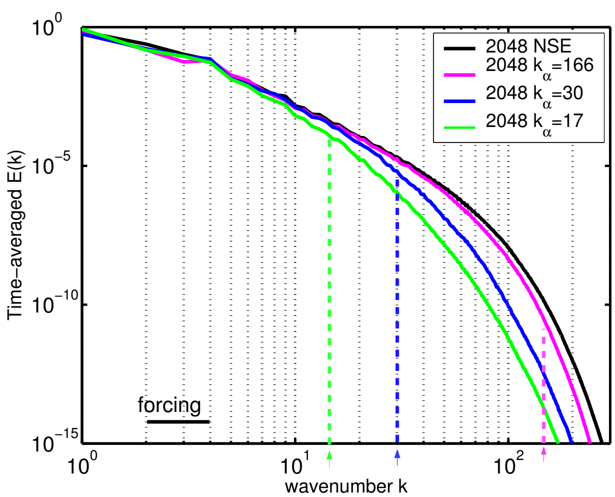


*Figure 2: A log-log plot of the energy spectrum for a* $2048^2$ *grid simulation. The 2D NS-*$\alpha$ *model can reproduce most of the large scale features of turbulent flows (those scales of motion corresponding to wave numbers less than* $k_\alpha$*).*

We next compute the DNS at a higher resolution and the 2D NS-$\alpha$ model at a lower resolution keeping the viscosity fixed and analyze the spectra for the large scales (in particular those scales greater than the length-scale $\alpha$). The result in Figure 3 suggests that the simulation on the coarser grid using 2D-NS-$\alpha$ (with nonzero alpha) can capture the statistics of the higher resolution

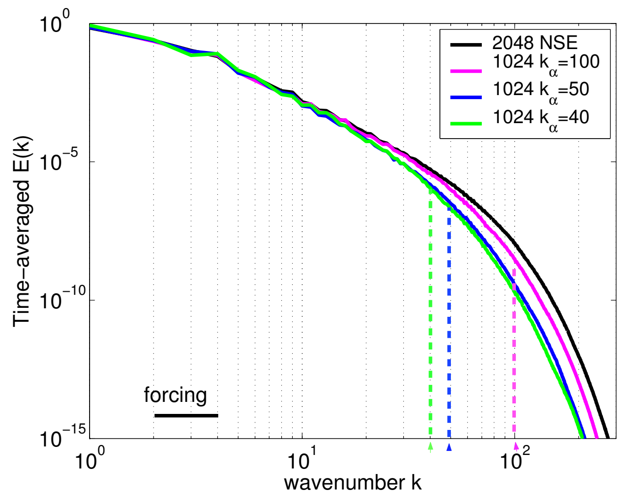


*Figure 3: A log-log plot of the energy spectrum. The 2D NS-*$\alpha$ *model can reproduce most of the large scale features of turbulent flows at a coarser grid. Here the viscosity is fixed. We see that running the 2D NS-*$\alpha$ *at a coarser grid* $(1024^2)$ *captures most of the large scale statistics of the more refined* $(2048^2)$ *DNS (black).*

DNS for most of the scales of motion greater than the length-scale $\alpha$. This shows that the NS-$\alpha$ is computationally cheaper than the NSE for calculating turbulent flows.

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## References

[1] C. Foias, D.D. Holm and E.S. Titi, *The three dimensional viscous Camassa–Holm equations, and their relation to the Navier-Stokes equations and turbulence theory,* J. Dynam. Differential Equations **14** (2002), 1–35.

[2] C. Foias, D.D. Holm and E.S. Titi, *The Navier-Stokes-alpha model of fluid turbulence. Advances in nonlinear mathematics and science,* Phys. D **152/153** (2001), 505–519.

[3] D. Holm, C. Jeffrey, S. Kurien, D. Livescu, M. Taylor, and Beth Wingate, *The LANS-*$\alpha$ *model for computing turbulence,* Los Alamos Science No. 29 (2005), 152–171.